7.3.4 Autocorrelation Functions for Random Pulse Trains

As another example of calculating autocorrelation functions, consider a random process with sample functions that can be expressed as

$$X(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT - \Delta) \tag{7.49}$$

where $\ldots a_{-1}, a_0, a_1, \ldots, a_k \ldots$ is a doubly-infinite sequence of random variables with

$$E[a_k a_{k+m}] = R_m \tag{7.50}$$

The function $p(t)$ is a deterministic pulse-type waveform where T is the separation between pulses; Δ is a random variable that is independent of the value of a_k and uniformly distributed in the interval $(-T/2, T/2)$.[5] The autocorrelation function of this waveform is

$$\begin{aligned} R_X(\tau) &= E[X(t)\,X(t+\tau)] \\ &= E\left\{ \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k a_{k+m} p(t - kT - \Delta)\, p\,[t + \tau - (k+m)T - \Delta] \right\} \end{aligned} \tag{7.51}$$

Taking the expectation inside the double sum and making use of the independence of the sequence $\{a_k a_{k+m}\}$ and the delay variable Δ, we obtain

$$\begin{aligned} R_X(\tau) &= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\left[a_k a_{k+m}\right] E\left\{p(t - kT - \Delta)\, p\,[t + \tau - (k+m)T - \Delta]\right\} \\ &= \sum_{m=-\infty}^{\infty} R_m \sum_{k=-\infty}^{\infty} \int_{-T/2}^{T/2} p(t - kT - \Delta)\, p\,[t + \tau - (k+m)T - \Delta]\, \frac{d\Delta}{T} \end{aligned} \tag{7.52}$$

The change of variables $u = t - kT - \Delta$ inside the integral results in

$$\begin{aligned} R_X(\tau) &= \sum_{m=-\infty}^{\infty} R_m \sum_{k=-\infty}^{\infty} \int_{t-(k+1/2)T}^{t-(k-1/2)T} p\,(u)\, p\,(u + \tau - mT)\, \frac{du}{T} \\ &= \sum_{m=-\infty}^{\infty} R_m \left[\frac{1}{T} \int_{-\infty}^{\infty} p\,(u + \tau - mT)\ p\,(u)\ du \right] \end{aligned} \tag{7.53}$$

Finally we have

$$R_X(\tau) = \sum_{m=-\infty}^{\infty} R_m\, r\,(\tau - mT) \tag{7.54}$$

where

$$r\,(\tau) \triangleq \frac{1}{T} \int_{-\infty}^{\infty} p\,(t + \tau)\, p\,(t)\ dt \tag{7.55}$$

is the pulse-correlation function. We consider the following example as an illustration.

EXAMPLE 7.6

In this example we consider a situation where the sequence $\{a_k\}$ has memory built into it by the relationship

$$a_k = g_0 A_k + g_1 A_{k-1} \tag{7.56}$$

where g_0 and g_1 are constants and the A_k's are random variables such that $A_k = \pm A$ where the sign is determined by a random coin toss independently from pulse to pulse for all k (note that if $g_1 = 0$, there is no memory). It can be shown that

$$E[a_k a_{k+m}] = \begin{cases} \left(g_0^2 + g_1^2\right) A^2, & m = 0 \\ g_0 g_1 A^2, & m = \pm 1 \\ 0, & \text{otherwise} \end{cases} \tag{7.57}$$

The assumed pulse shape is $p(t) = \Pi\left(\frac{t}{T}\right)$ so that the pulse-correlation function is

$$\begin{aligned} r(\tau) &= \frac{1}{T}\int_{-\infty}^{\infty} \Pi\left(\frac{t+\tau}{T}\right)\Pi\left(\frac{t}{T}\right) dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \Pi\left(\frac{t+\tau}{T}\right) dt = \Lambda\left(\frac{\tau}{T}\right) \end{aligned} \tag{7.58}$$

Thus, the autocorrelation function (7.58) becomes

$$R_X(\tau) = A^2\left\{\left[g_0^2 + g_1^2\right]\Lambda\left(\frac{\tau}{T}\right) + g_0 g_1\left[\Lambda\left(\frac{\tau+T}{T}\right) + \Lambda\left(\frac{\tau-T}{T}\right)\right]\right\} \tag{7.59}$$

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